Individual Results

-- First Place --
Aneesh Dasgupta, Carmel High School
Nicholas Gao, Hamilton Southeastern High School

-- Second Place --
Ahmed Abdalla, Avon High School
Canaan He, Carmel High School
Bill Qian, Carmel High School
Skylar Smith, Fishers High School
William Teasley, Cathedral High School

-- Third Place --
Dillon Blevins, Avon High School
Pravir Chugh, Park Tudor
Jui Desai, Avon High School
Harsh Duvvuru, Avon High School
Akiharu Fukuda, Hamilton Southeastern High School
Prabhvir Lakhon, Avon High School
Arnab Mehra, Hamilton Southeastern High School
Samantha Miller, Avon High School
Connor O’Neill, Avon High School
Collin Tully, Fishers High School

Team Results

-- First Place --
TS336, Zionsville Community High School: Tate Eugenio, Alex Michie, Ethan Perry & Ryan Walter

-- Second Place --
hhhh ddd hd d hdhh hdd d dh, Center Grove High School: Mahek Agrawal, Vinay Bhamidipati, Carter Dills & Josh Stevenson

-- Third Place --
Calcoholics, Center Grove High School: Rahul Appaji, Lucas Embrey & Samuel Wong

-- Honorable Mention --
Just Collin, Fishers High School: Collin Tully

Morgan Grace Browne, Zionsville Community High School: Adrianna Black, Beth Larsen & Wesley Turnbull

The AP Calc Bros, Center Grove High School: Erin Clingerman, Christina Monev, Sapna Vyas & Sonya Vyas

The LaGrange Errors, Center Grove High School: Sophia Hagedorn, Jake Miller, David Pham & Zane York

ZEAM Team, Zionsville Community High School: Matthew Crandall, Eli Everson, Zachary Herbon & Amelia Ridolfo

-- SCHOOL AWARD --
Avon High School, Teacher: Anthony Record

-- SPIRIT AWARD --
Center Grove High School, Teachers: Karen Fruits & John Moore

THANK YOU!
To all of the supportive and encouraging teachers:

Lisa Boyl
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Andra Edgell
Christina Ferguson
Lisa Ford
Karen Fruits
Rebecca Hufty
Daniel Mach
Letitia McCallister
John Moore
Anthony Record
Carmen Taylor

IUPUI
SCHOOL OF SCIENCE
Department of Mathematical Sciences
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Dillon Blevins, Avon High School
Danielle Boehm, Hamilton Southeastern High School
Samuel Chilson, Hamilton Southeastern High School
Sruthi Chintalacharuvu, Hamilton Southeastern High School
Pravir Chugh, Park Tudor
Samuel Crook, Avon High School
Yossra Daiya, Hamilton Southeastern High School
Aneesh Dasgupta, Carmel High School
Nuhamin Demeku, Avon High School
Jui Desai, Avon High School
Harsh Duvvuru, Avon High School
Eliza Eisenmann, Hamilton Southeastern High School
Victoria Elliot, Hamilton Southeastern High School
Akhilu Fukuda, Hamilton Southeastern High School
Katelynn Gallagher, Avon High School
Abby Gambrall, Avon High School
Nicholas Gao, Hamilton Southeastern High School
Chloe Graham, Hamilton Southeastern High School
Kyle Griffin, Center Grove High School
Jayne Harlan, Avon High School
Canaan He, Carmel High School
Chloe Helmreich, Hamilton Southeastern High School
Jack Herzog, Hamilton Southeastern High School
Emily Hodge, Hamilton Southeastern High School
Logan Hubbard, Hamilton Southeastern High School
Caroline Jackson, Avon High School
Suyash Johari, Fishers High School
Sriya Koganti, Avon High School
Prabhvir Lakhan, Avon High School
Ayoola Laleye, Avon High School
Jason Le, Avon High School
Andy Le, Avon High School
Elijah Loby, Southport High School
Jack McClanahan, Avon High School
Arnav Mehra, Hamilton Southeastern High School
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Emilia Myers, Avon High School
Nolan Natalie, Avon High School
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Mebele Onwuaduegbo, Avon High School
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Isaac Robbins, Hamilton Southeastern High School
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Gabriel Ruiz, Hamilton Southeastern High School
Peyton Schaefer, Avon High School
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Alicia Stoppel, Avon High School
Aniket Sundaresh, Hamilton Southeastern High School
William Teasley, Cathedral High School
Collin Tully, Fishers High School
Macy Walters, Hamilton Southeastern High School
Kevin Wang, Carmel High School
Caslyn Whitesell, Avon High School

Algebread, Center Grove High School: Samuel Howe, Bryce Rayner
Calcoholics, Center Grove High School: Rahul Appaji, Lucas Embrey & Samuel Wong
Caroline & Matthew, Avon High School: Matthew Clupper & Caroline Jackson
Gabriel Ruiz, Hamilton Southeastern High School: Garbriel Ruiz
hhhh ddd hd d hdhh d dh, Center Grove High School: Mahek Agrawal, Vinay Bhamidipati, Carter Dills & Josh Stevenson
JAJ, Avon High School: Abby Gambrall, Jack McClanahan & Jazmynn Peterson
Jayne & Braeden, Avon High School: Jayne Harlan & Braeden Record
Just Collin, Fishers High School: Collin Tully
Kailyn & Bella’s Team, Center Grove High School: Bella Ferris & Kailyn Spitler
Logan Hubbard, Hamilton Southeastern High School: Logan Hubbard
Morgan Grace Browne, Zionsville Community High School: Adrianna Black, Beth Larsen & Wesley Turnbull
Oriole Math Squad, Avon High School: Katelynn Gallagher & Samantha Miller
SHIELD, Avon High School: Emilia Myers, Taylor Rueff & Anthony Whittle
Sophia & Sriya, Avon High School: Sophia Ko & Sriya Koganti
Team 1, Fishers High School: Suyash Johari, Skylar Thompson
Team SPV, Center Grove High School: Parthiv Patel, Vaibhata Potturu & Sahil Sura
The A Team, Center Grove High School: Kaia Hunter, Katelyn Jansen & Jillian Ransdell
The Algebros, Avon High School: Darian Rostam & Kevin Rostam
The AP Calc Bros, Center Grove High School: Erin Clingerman, Christina Monev, Sapna Vyas & Sonya Vyas
The LaGrange Errors, Center Grove High School: Sophia Hagedorn, Jake Miller, David Pham & Zane York
TS336, Zionsville Community High School: Tate Eugenio, Ethan Perry, Alex Michie & Ryan Walter
We Tried, Center Grove High School: Sarah Pack & Mackenzie Souchon
ZEAM Team, Zionsville Community High School: Matthew Crandall, Eli Everson, Zachary Herbon & Amelia Ridolfo
Problem 1

Two values, $x$ and $y$ are selected at random and independently of each other from the interval $[0, 1]$ with all the outcomes being equally likely. What is the probability that $|x - y| > 1/2$?

An equivalent question asks what is the probability that $|x - y| > 1/2$ when a point $(x, y)$ is selected at random from the square $[0, 1] \times [0, 1]$ with all the outcomes being equally likely. All the outcomes being equally likely means that the probability is proportional to the area of subregion of the square $[0, 1] \times [0, 1]$ where $|x - y| > 1/2$, that is, where either $y - x > 1/2$ or $y - x < -1/2$. This subregion consists of two isosceles right triangles with legs of length $1/2$. Hence, the area of each triangle is $1/8$ and the total area is $1/4$. Since the area of the square is 1, the probability is $1/4$ as well.
**Problem 2**

Let $B$ be a “mysterious blob”. If you hold $B$ under the sun and see a perfectly round shadow, $B$ does not need to be a perfectly round ball. For example, it could be a cylinder that is held perfectly vertically. Suppose that you hold $B$ under the sun and see a perfectly round shadow and then rotate $B$ to a new orientation and again see a perfectly round shadow of the same diameter. Must $B$ be a perfectly round ball? What if you are able to rotate $B$ twice, getting now two new orientations, and all three shadows that you see are perfectly round and of the same diameter. Must $B$ be a ball?

Without loss of generality, suppose that each of the shadows is a disc of radius 1.

Notice that having one orientation at which the blob has a perfectly round shadow of radius 1 is equivalent to being able to find a hollow cylinder of radius 1 such that it “perfectly fits” over the blob. That is, any line parallel to the axis of the cylinder that is inside of the cylinder must hit the blob and any such line outside of the cylinder must not hit the blob.

As explained in the statement of the problem, with only one such perfectly round shadow, $B$ could be a solid cylinder whose axis is parallel to that of the hollow cylinder mentioned above.

Having two perfectly round shadows of radius 1 corresponds to having two different hollow cylinders that “fit perfectly” over $B$. Suppose that we take a perfectly round ball of radius 1. Then, each of these two hollow cylinders only hits the surface of the ball on a “great circle”. Therefore, making a small indentation to the ball away from these two great circles will not affect the condition that both cylinders fit perfectly over $B$. Therefore, with two perfectly round shadows $B$ need not be a perfectly round ball.
The same holds if there are 3 (or any finite number $k$) of different orientations of $B$ which result in perfectly round shadows. Each corresponds to a hollow cylinder of radius 1 that fits perfectly around the blob. If you take a perfectly round ball, each of them will intersect its boundary in a great circle. One can then make a small indent in the ball away from these finitely many great circles and that will not affect the $k$ different perfectly round shadows. Therefore, with $k$ perfectly round shadows, $B$ need not be a perfectly round ball.

It turns out that if every possible shadow is perfectly round, then the mysterious blob is a perfectly round ball. See Problem 3 from the 2014 IUPUI High School Math Contest: https://math.iupui.edu/sites/default/files/2014_hsmc.pdf.
Problem 3

Show that \( \sum_{n=1}^{\infty} \frac{1}{4n^4 + 16n^3 + 23n^2 + 14n + 3} = \frac{10 - \pi^2}{6} \).

Observe that

\[ 4n^4 + 16n^3 + 23n^2 + 14n + 3 = (n + 1)^2(2n + 1)(2n + 3). \]

Therefore,

\[ \frac{1}{4n^4 + 16n^3 + 23n^2 + 14n + 3} = -\frac{1}{(n + 1)^2} + \frac{2}{2n + 1} - \frac{2}{2n + 3}. \]

That is, the original sum is equal to

\[ \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} + \sum_{n=1}^{\infty} \left( \frac{2}{2n + 1} - \frac{2}{2n + 3} \right). \]

Clearly,

\[ \sum_{n=1}^{\infty} \left( \frac{2}{2n + 1} - \frac{2}{2n + 3} \right) = \left( \frac{2}{3} - \frac{2}{5} \right) + \left( \frac{2}{5} - \frac{2}{7} \right) + \left( \frac{2}{7} - \frac{2}{9} \right) + \cdots = \frac{2}{3} \]

and, see https://en.wikipedia.org/wiki/Basel_problem,

\[ \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = -1 + \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = -1 + \frac{\pi^2}{6}, \]

Thus,

\[ \sum_{n=1}^{\infty} \frac{1}{4n^4 + 16n^3 + 23n^2 + 14n + 3} = 1 - \frac{\pi^2}{6} + \frac{2}{3} = \frac{10 - \pi^2}{6}. \]
Problem 4

Let $p_n(x)$ be a polynomial of degree $n$ such that $p(0) = 1$, $p(1) = a$, $p(2) = a^2, \ldots, p(n) = a^n$ for some number $a$. What is $p(n+1)$?

If $a = 1$, then $p_n(x) \equiv 1$. So, assume that $a \neq 1$. Let's write this polynomial as

$$p_n(x) = c_0 + c_1(x-n) + c_2(x-n+1)(x-n) + \cdots + c_n(x-1) \cdots (x-n).$$

Clearly, $a^n = p(n) = c_0$ and $a^{n-1} = p(n-1) = c_0 - c_1$. Therefore, $c_1 = -a^{n-1}(1-a)$. Continuing in the same manner one can guess that

$$c_k = (-1)^k \frac{a^n-k(a-1)^k}{k!} = \frac{a^n-k(a-1)^k}{k!}.$$

Indeed, when $0 \leq m \leq n$, it holds that

$$p(m) = c_0 + (m-n)c_1 + (m-n+1)(m-n)c_2 + \cdots + (-1)^k(m-k)\cdots(m-n)\cdot c_{n-m}$$

$$= c_0 - \frac{(n-m)!}{(n-m-1)!} c_1 + \frac{(n-m)!}{(n-m-2)!} c_2 + \cdots + (-1)^n \frac{(n-m)!}{1!} c_{n-m}.$$

Plugging in the guessed expression for $c_k$'s gives

$$p(m) = a^n + \frac{(n-m)!}{(n-m-1)!} a^{n-1}(1-a) + \frac{(n-m)!}{(n-m-2)!} a^{n-2}(1-a)^2 + \cdots + \frac{(n-m)!}{1!} (1-a)^{n-m}$$

$$= a^n \left( \binom{n-m}{0} a^{n-m} + \binom{n-m}{1} a^{n-m-1}(1-a) + \binom{n-m}{2} a^{n-m-2}(1-a)^2 + \cdots + \binom{n-m}{n-m} (1-a)^{n-m} \right)$$

$$= a^n (a+1-a)^{n-m} = a^n.$$

Similarly, it holds that

$$p(n+1) = c_0 + 1! c_1 + 2! c_2 + \cdots + n! c_n$$

$$= a^n + a^{n-1}(a-1) + a^{n-2}(a-1)^2 + \cdots + (a-1)^n$$

$$= a^n \left( 1 + (1-1/a) + (1-1/a)^2 + \cdots + (1-1/a)^n \right)$$

$$= a^n \frac{1 - (1-1/a)^{n+1}}{1 - (1-1/a)} = a^{n+1} - (a-1)^{n+1}.$$

Therefore, $p(n+1) = a^{n+1} - (a-1)^{n+1}$ (for all $a$).
Team Problem

This is based on an interesting problem I discovered which is isomorphic to “Given a set S which is partitioned into to two subsets A(S) and A'(S), you may select a subset of S and ask how many elements of that particular subset are in A(S). How many questions are necessary to determine which elements of S are in A(S) and which are not?”

Which leads to a set of linear equations in which each variable has both coefficient of 0 or 1 and value of 0 or 1, but the right-hand side faces no such restriction.

As such, this is how the solution will read, rather than the original problem (which is different only because the right-hand side is an average, rather than a sum).

(A) There are two non-isomorphic ways to resolve 4 variables in 3 equations. One, as noted is

\[
\begin{align*}
A + B + C &= W_1 \\
A + B + D &= W_2 \\
A + C + D &= W_3
\end{align*}
\]

Which leads to \(3A + 2B + 2C + D = W_1 + W_2 + W_3\), from which we can determine the parity of A and then have 3 independent equations in three unknowns.

The other is \(A + B + C = W_1\)

\[
\begin{align*}
A + D &= W_2 \\
C + D &= W_3
\end{align*}
\]

And if we add the first two equations and subtract the third, we get \(2A + B = W_1 + W_2 - W_3\), which can be treated as a binary number and gives us A and B at once. C and D follow trivially.

(B) There are presumably many ways to resolve 7 variables in 5 equations. Arguably, the simplest builds off of the solution to (A), as follows:

\[
\begin{align*}
A + B + C + D &= W_1 \\
A + B + E &= W_2 \\
A + F &= W_3 \\
A + G &= W_4 \\
D + E + F + G &= W_5
\end{align*}
\]

Now adding the first four and subtracting the 5th leads to \(4A + 2B + C = W_1 + W_2 + W_3 + W_4 - W_5\), which can be treated as a binary number and gives us A, B, and C at once. D, E, F, and G follow trivially.

(C) Resolving 9 variables in 6 equations is a little bit trickier. Again, I’m sure that there are many ways. One that is intuitive uses the same basic idea as the previous two, with a twist.

\[
\begin{align*}
A + B + E + F &= W_1
\end{align*}
\]
\[
\begin{align*}
A + C + G + H &= W_2 \\
A + G + I &= W_3 \\
A + B + D + E + H + I &= W_4 \\
F + H + I &= W_5 \\
E + F + G + H + I &= W_6
\end{align*}
\]

It's not easy to see what this accomplishes at first, until you divine that it is designed so that the following sum reduces to a binary as before: \(8A + 4B + 2C + D = 3W_1 + 2W_2 + 2W_3 + W_4 + W_5 - 4W_6\), and once \(A, B, C,\) and \(D\) have been resolved, it is easy to solve for \(E, F, G, H,\) and \(I\) \((I = W_6 - (W_1 - A - B) - (W_2 - A - C))\), for example, and from here you can use the third equation to find \(G\), then the second to find \(H\), and the fourth and fifth to find \(E\) and \(F\).

One note that will matter in part \((D)\) is that 8, 4, 2, 1 are not the only set of potential coefficients with 16 distinct potential sums. Obvious large examples would include, say, 1000, 100, 10 and 1. A small example is 7, 6, 5, 3 – the largest of these is actually smaller than 8, which makes resolving 11 variables in 7 equations tractable.

\((D)\) My solution for resolving 11 variables in 7 equations takes advantage of the fact that the sums of all distinct subsets of \(\{13, 12, 11, 9, 6\}\) have distinct sums – 32 possibilities in all. Proof by exhaustion is probably easiest, and left to the reader.

\[
\begin{align*}
A + B + C + D + E + F &= W_1 \\
A + B + C + D + G + H &= W_2 \\
A + B + C + D + E + I + J &= W_3 \\
A + B + C + D + E + I + K &= W_4 \\
A + B + C + D + E + J + K &= W_5 \\
A + B + C + D + E + J + K &= W_6 \\
F + G + H + I + J + K &= W_7
\end{align*}
\]

Now \(13A + 12B + 11C + 9D + 6E = 4W_1 + 3W_2 + 2W_3 + 2W_4 + W_5 + W_6 - 4W_7\), and again, after resolving \(A, B, C, D,\) and \(E\) and subtracting from the right-hand side of the equations, we are left with 7 dependent equations in 6 variables (the first six of which are independent and can be solved). Indeed, you may notice that the system for \(G, H, I, J,\) and \(K\) is identical to the system for \(E, F, G, H,\) and \(I\) from part \((C)\). Obviously, \(F = W_1 - A - B - C - D - E\).

\((E)\) I came up with my solution for resolving 14 variables in 8 equations before I solved parts \((C)\) and \((D)\), actually, so it may look a little simpler. It exploits the same logic used in these, but it also
exploits a method of resolving 10 variables in 7 equations embedded in it with the property that all of the columns of the latter have coefficients that add up to 4.

\[
A + B + C + D + E + F + H + I + J + K = W_1
\]
\[
A + B + C + D + E + F + H + L + M + N = W_2
\]
\[
A + B + C + D + E + G + H + I + J + L = W_3
\]
\[
A + B + C + D + E + G + I + K + M + N = W_4
\]
\[
A + B + C + D + F + G + J + N = W_5
\]
\[
A + B + D + F + G + K + L + M = W_6
\]
\[
A + D + H + I + J + K + L + M + N = W_7
\]
\[
D + E + F + G + H + I + J + K + L + M + N = W_8
\]

Now \(7A + 6B + 5C + 3D = W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 - 4W_8\). In this case, untangling the other equations is a little trickier. Also, astute observers may wonder why \(D\) appears in all 8 equations, when it could equally well appear in any 3 of the first 7. That is a valid question about this solution, but the reason why \(I\) had \(D\) appear 8 times here will become clearer in part (F).

At any rate, while it’s not a simple binary number, it turns out that the subsets of \(\{3, 5, 6, 7\}\) all have different sums, so \(A, B, C,\) and \(D\) can be resolved. This is the first case in which all subsets of the set \(S_n = \{a_1, a_2, a_3, \ldots a_n\}\) have different sums, the elements of \(S_n\) are all positive integers in ascending order, and \(a_n < 2^n\). It is not, however, the last (although by the Pigeonhole Principle it is easy to show that \(a_n > 2^n/n\) no matter what \(n\) is).

As before, we have shed ourselves of some variables and have a set of dependent equations. We now work with he subsystem

\[
E + F + H + I + J + K = X_1
\]
\[
E + F + H + L + M + N = X_2
\]
\[
E + G + H + I + J + L = X_3
\]
\[
E + G + I + K + M + N = X_4
\]
\[
F + G + J + N = X_5
\]
\[
F + G + K + L + M = X_6
\]
\[
H + I + J + K + L + M + N = X_7
\]

where \(X_i\) is the difference between \(W_i\) and whichever of \(A, B, C,\) and \(D\) appear on the left-hand side.

Now note that
\[ X_1 + X_2 - X_3 = 2E + 2F + H \]
\[ X_3 + X_4 - X_5 = 2E + 2G + I, \text{ and} \]
\[ X_1 + X_2 + X_3 + X_4 + X_5 + X_6 - 3X_7 = 4E + 4F + 4G \]

The first two allow us to resolve H and I by parity, and we are then left with three independent equations in E, F, and G, allowing us to resolve them as well.

After resolving these, adjusting the right-hand sides, and eliminating the 4th, 5th and 7th equations as redundant, we have

\[
\begin{align*}
J + K &= Y_1 \\
L + M + N &= Y_2 \\
J + L &= Y_3 \\
K + L + M &= Y_6
\end{align*}
\]

Now note that the 1st, 3rd, and 6th equations mirror those of part (A), allowing us to resolve J, K, L, and M, and resolving N at this point using the second equation is a triviality.

(F) I will show how to resolve 48 variables with 27 equations. Doing this twice and resolving an additional 4 variables with 3 equations will lead to resolving 100 variables in 57 equations. I rather doubt that this is the best possible solution; it is, however, the best I have come up with. A similar technique would lead one to a means of resolving 91 variables with 51 equations, and then implementing an additional 6 equations to resolve 9 variables accomplishes the same thing.

First, I will start with three of what I will call an 8-by-14 block, as above. They will be set up diagonally, as so:

<table>
<thead>
<tr>
<th>6 columns</th>
<th>14 columns</th>
<th>14 columns</th>
<th>14 columns</th>
<th>Right-Hand Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 Rows</td>
<td>Designed to sum to unique</td>
<td>Block 1</td>
<td>All zeroes</td>
<td>All zeroes</td>
</tr>
<tr>
<td>8 Rows</td>
<td>Designed to sum to unique</td>
<td>All zeroes</td>
<td>Block 2</td>
<td>All zeroes</td>
</tr>
<tr>
<td>8 Rows</td>
<td>Designed to sum to unique</td>
<td>All zeroes</td>
<td>All zeroes</td>
<td>Block 3</td>
</tr>
<tr>
<td>Binary 1-digit</td>
<td>000000</td>
<td>10101111111111</td>
<td>10101111111111</td>
<td>10101111111111</td>
</tr>
<tr>
<td>Binary 2-digit</td>
<td>000000</td>
<td>01101111111111</td>
<td>01101111111111</td>
<td>01101111111111</td>
</tr>
</tbody>
</table>
The coefficients in the Binary 1-digit row and 2-digit row each consist of 6 ‘0’s followed by three identical blocks of 14 coefficients in order. The 2-digit row has \{0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}, while the 1-digit row has \{1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}. By design, they are almost identical. The 8*Level Row has 6 ‘0’s followed by 42 ‘1’s.

Now, temporarily ignoring the first six columns, observe what happens when we add the first 24 rows to the 25th row and double the 26th row: within each block, the last 10 variables appear 5 times each, and then we add 1 + 2 to the coefficient to get 8. The first four variables appear between 5 and 8 times, but the design of the binary 1-digit and 2-digit rows brings the total of these coefficients up to 8 as well (this is WHY I included D in every row in part (E).) Thus we can now subtract 8 times the final row and all of these variables vanish.

This isolates the first 6 variables. We deliberately select coefficients in the top 24 rows such that when we add them together we have 24A + 23B + 22C + 20D + 17E + 11F on the left. As the reader can no doubt easily prove, the sums of the subsets of \(S_6 = \{11, 17, 20, 22, 23, 24\}\) are all distinct. We thus resolve the first six variables and then are left with three distinct 14-by-8 systems which we have already proven we can resolve.

Incidentally, another thing the reader might enjoy exploring is how to generate the following sets systematically – they have subsets with distinct sums, and for two subset A and B, \(N(A) < N(B)\), then the sum of the elements of A is less than the sum of the elements of B:

\[ S_1 = \{1\}, S_2 = \{1, 2\}, S_3 = \{2, 3, 4\}, S_4 = \{3, 5, 6, 7\}, S_5 = \{6, 9, 11, 12, 13\}, S_6 = \{11, 17, 20, 22, 23, 24\}, \ldots \]

Curiously, \(S_7 = \{20, 31, 37, 40, 42, 43, 44\}\) lacks that final property \((20+31+37+40 < 42+43+44)\). However, \(S^*_7 = \{22, 33, 39, 42, 44, 45, 46\}\) has that last property along with distinct subset-sums.